NUMERICAL METHOD OF SOLVING HEAT CONDUCTION PROBLEMS FOR BODIES OF COMPLEX GEOMETRY

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A grid method of solving nonstationary problems of heat conduction for bodies with curvilinear boundaries is proposed. The method is based on approximating the heat-transfer equation by a balance equation for a canonical shape element, constructed on a nonuniform difference grid.

The necessity of introducing nonuniform difference grids arises in numerical simulation of physical fields in bodies of complex geometry with curvilinear boundaries. One of three approaches is usually applied to obtain in this case equations approximating the corresponding differential equations. The first approach is related to constructing finite-difference equations on nonuniform grids by combining series expansions of the unknown functions for nodal points located in the vicinity of the given nodal point [1]. The second approach is related to that exposed [2] on the considered region of a uniform difference grid. Though in this case values of the required function are determined on the nonuniform grid only at near-boundary and boundary nodal points, nevertheless the algorithm is quite complex in its specific procedures of calculating functions for different nodal points of the region. The third approach is related to using the finite element method [3]. Its implementation algorithm is complicated, and estimating the error in the approximate equations used in this case involves certain difficulties.

In the following we describe a method of solving heat conduction problems for bodies of complex geometry, whose implementation algorithm on a computer does not differ substantially from the algorithm of solving heat conduction problems for canonical shape bodies. The method is based on approximating the transport equation by a balance equation for a canonical shape element, constructed on a nonuniform difference grid.

Consider an arbitrary region G of finite size in a p-dimensional orthogonal coordinate system. For the sake of simplicity, the detailed discussion of the method, which we call the canonical element method, is provided for nonstationary problems of heat conduction for a two-dimensional ($p = 2$) singly-connected region in a Cartesian coordinate system (x, y). Let y' and y'' be the minimum and maximum values of the coordinate y for points of the region G. We construct a family of coordinate lines

$$y_m = y_{m-1} + \Delta y_m,\ m = 1, 2 \ldots, M,\ y_0 = y',\ y_M = y''. \tag{1}$$

We denote by x_m' and x_m'' the minimum and maximum values of the coordinate x for points of the body located on the coordinate line $y = y_m$. On the segment $x_m' < x < x_m''$, $m = 0, 1, \ldots, M$, we introduce the family of nodal points

$$x_{im} = x_{i-1,m} + \Delta x_{im},\ i = 0, 1, \ldots, I,\ x_{0,m} = x_m',\ x_{Im} = x_m''. \tag{2}$$

We note that the points (x_{i0}, y_0), (x_{iM}, y_M), $(X_{0,m}, y_m)$, (x_{Im}, y_m) belong to the boundary surface of the body under consideration. The time coordinate τ is divided by planes

$$\tau_n = \tau_{n-1} + \Delta\tau_n,\ n = 0, 1, \ldots, \Delta\tau_n > 0. \tag{3}$$

To construct the difference equation, approximating the heat conduction equation on the grid (1)-(3) we write down the energy balance equation for the canonical shape element (rectangle) of the region G (see Fig. 1) formed by the coordinate planes $(y_{m+1} + y_m)/2$, $(y_m + y_{m-1})$ 2, $(x_{i+1,m} + x_{im})/2$, $(x_{im} + x_{i-1,m})/2$.

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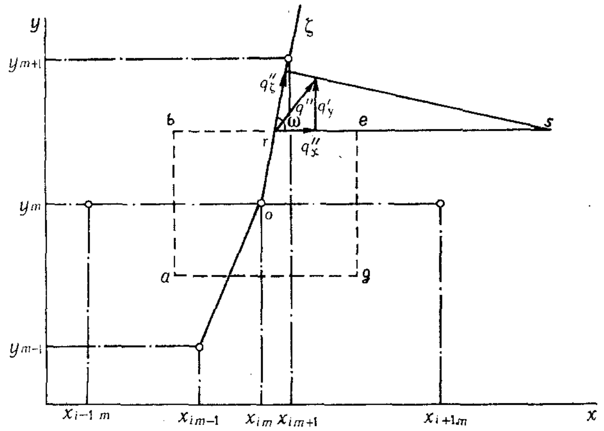

Fig. 1. Fragment of nonuniform difference grid with canonical shape element abeg.

The specific thermal fluxes through the surfaces eg and ba at the temporal layer n are found, with an error of order $O(x_{im}^2)$, by the relations

$$q''_{xim} = -\frac{\lambda_{i+1,m} + \lambda_{im}}{2}\,\frac{t^n_{i+1,m} - t^n_{im}}{\Delta x_{i+1,m}}, \tag{4}$$

$$q'_{xim} = \frac{\lambda_{im} + \lambda_{i-1,m}}{2}\,\frac{t^n_{im} - t^n_{i-1,m}}{\Delta x_{im}}. \tag{5}$$

The specific thermal flux through the surface can be calculated in terms of values of the specific thermal fluxes $\bar{q}_x''$ and q_ζ'' along the surface be and along the 0ζ-axis. We note that $\bar{q}_x''$ and q_ζ'' are the projections of the maximum thermal flux q" on the x- and 0ζ-axes, while the projection of q" on the y-axis is the specific thermal flux q_y''. Flux $\bar{q}_x''$ is determined at $\theta_q = 0.5$, with an error of order $O(\Delta y_m^2 + \Delta x_{im}^2)$, by the difference expression

$$\bar{q}''_x = (1-\theta_q)\,\frac{q'_{xim} + q''_{xim}}{2} + \theta_q\,\frac{q'_{xim+1} + q''_{xim+1}}{2}, \tag{6}$$

where $0 \le \theta_q \le 0.5$. If the angle $\pi/2 - \omega$ between the y- and 0ζ-axes is relatively small, without loss of solution accuracy one may put $\theta_q = 0$. With an error of order $(\Delta x_{im}^2 + \Delta y_m^2)$, the flux q_ζ'' is calculated by the relation

$$q''_\zeta = -\frac{\lambda_{im+1} + \lambda_{im}}{2}\,\frac{t^n_{im+1} - t^n_{im}}{\sqrt{(y_{m+1} - y_m)^2 + (x_{im+1} - x_{im})^2}}. \tag{7}$$

It follows from Fig. 1 that $rs = q_\zeta''/\sin(\pi/2 - \omega)$, $us = rs - q_x'' = q_\zeta''/\sin(\pi/2 - \omega) - q_x''$, $q_y'' = us\,\mathrm{tg}(\pi/2 - \omega)$, or

$$q''_y = \frac{q''_\zeta}{\cos\left(\dfrac{\pi}{2} - \omega\right)} - \bar{q}''_x\,\mathrm{tg}\left(\frac{\pi}{2} - \omega\right), \tag{8}$$

where ω is the angle between the x- and 0ζ-axes. Since

$$\cos\left(\frac{\pi}{2} - \omega\right) = \frac{y_{m+1} - y_m}{\sqrt{(y_{m+1} - y_m)^2 + (x_{im+1} - x_{im})^2}},$$

$$\mathrm{tg}\left(\frac{\pi}{2} - \omega\right) = \frac{x_{im+1} - x_{im}}{y_{m+1} - y_m},$$

expression (8) can be written in the form

$$q''_y = -\frac{\lambda_{im+1} + \lambda_{im}}{2}\,\frac{t^n_{im+1} - t^n_{im}}{y_{m+1} - y_m} - \bar{q}''_x\,\frac{x_{im+1} - x_{im}}{y_{m+1} - y_{im}}; \tag{9}$$

the specific thermal flux q_y' through the surface *ag* is determined similarly to (9) by the relation

$$q_y' = -\frac{\lambda_{im}+\lambda_{im-1}}{2}\frac{t_{im}^n - t_{im-1}}{y_m - y_{m-1}} - \bar{q}_x' \frac{x_{im}-x_{im-1}}{y_m - y_{m-1}}, \tag{10}$$

where

$$\bar{q}_x' = (1-\theta_q)\frac{q_{xim}' + q_{xim}''}{2} + \theta_q \frac{q_{xim-1}' + q_{xim-1}''}{2}.$$

According to the three-layer explicit difference scheme [4, 5], the energy balance equation can be written as follows:

$$c\rho\left[(1+\theta)\frac{t_{im}^{n+1}-t_{im}^n}{\Delta\tau^{n+1}} - \theta\frac{t_{im}^n - t_{im}^{n-1}}{\Delta\tau^n}\right] = \frac{q_x' - q_x''}{0{,}5\,(x_{im+1}-x_{im-1})} + \frac{q_y' - q_y''}{0{,}5\,(y_{m+1}-y_{m-1})}. \tag{11}$$

Here the parameter $\theta \geq 0$. For the case of constant values of λ, $\Delta y_m = \Delta y$, $\Delta x_{im} = \Delta x_m$, $\Delta\tau^n = \Delta\tau$ Eq. (11) acquires the form

$$(1+\theta)\frac{t_{im}^{n+1}-t_{im}^n}{\Delta\tau} - \theta\frac{t_{im}^n - t_{im}^{n-1}}{\Delta\tau} = \frac{\lambda}{c\rho}\left[\frac{t_{i+1,m}^n + t_{i-1,m}^n - 2t_{im}^n}{\Delta x_m^2} + \right.$$

$$+ \frac{t_{im+1}^n + t_{im-1}^n - 2t_{im}^n}{\Delta y^2} + (1-\theta_q)\frac{t_{i+1,m}^n - t_{i-1,m}^n}{2\Delta x}\,\frac{x_{im+1}+x_{im-1}-2x_{im}}{\Delta y^2} + \tag{12}$$

$$\left. + \theta_q\left(\frac{t_{i+1,m+1}^n - t_{i-1,m+1}^n}{2\Delta x_{m+1}}\,\frac{x_{im+1}-x_{im}}{\Delta y^2} - \frac{t_{i+1,m-1}^n - t_{i-1,m-1}^n}{2\Delta x_{m-1}}\,\frac{x_{im}-x_{im-1}}{\Delta y^2}\right)\right].$$

The necessary stability conditions of the difference equation (12) are

$$\frac{2\Delta\tau\lambda}{c\rho(1+2\theta)}\left(\frac{1}{\Delta x_m^2} + \frac{1}{\Delta y^2}\right) \leqslant 1, \tag{13}$$

$$\frac{x_{im+1}+x_{im-1}-2x_{im}}{\Delta y^2}\Delta x_m \leqslant 1. \tag{14}$$

Due to the possibility of varying the parameter θ, condition (13) practically imposes no restrictions on the step of the difference scheme. Condition (14) is satisfied if the surface of the body $x_\Gamma = x_\Gamma(y)$ is quite smooth, with $\partial^2 x_\Gamma(y)/\partial y^2\ \Delta x_m \leq 1$.

The temperature determination at the boundary nodal points (x_{om}, y_m) and (x_{Im}, y_m) does not lead to difficulties. If the heat transfer condition near the point (x_{Im}, y_m) is

$$A\frac{\partial t}{\partial \nu} + Bt + C = 0,$$

where ν is the direction of the normal to the boundary surface, to find the $t_{Im}{}^{n+1}$ values at this point one can use, for example, the following difference expression

$$A\frac{t_{Im}^{n+1} - t_{I-1,m}^{n+1}}{\Delta x_m \cos(x,\ \nu)} + Bt_{Im}^{n+1} + C = 0.$$

The calculation algorithm of the temperature field in a three-dimensional body of complex geometry by the canonical element method does not differ substantially from the algorithm of the two-dimensional problem. The construction of the difference grid in a three-dimensional body is advisably performed as follows. We construct a family of equivalent planes

$$z_j = z_j' + j\Delta z,\ j = 0,\ 1,\ 2,\ \ldots,\ J,\ \Delta z = \frac{z_j'' - z_j'}{J},$$

where z_j' and z_j'' are the maximum and minimum values of the coordinate z for points of the region considered. At each cross section of the body we introduce a family of equidistant planes: $y_{mj} = y_j' + m\Delta y_j$, $m = 0, 1, \ldots, M$, $\Delta y_j = (y_j'' - y_j')/M$, where y_j' and y_j'' are the minimum and maximum values of the coordinate y for points of the cross section z. Finally, at each segment of the line (z_j, y_{mj}) belonging to the body under consideration one introduces a family of nodal points

$$x_{imj} = x_{mj}' + i\Delta x_{mj},\ i = 0,\ 1,\ \ldots,\ I,\ \Delta x_{mj} = \frac{x_{mj}'' - x_{mj}'}{I},$$

where x_{mj}' and x_{mj}'' are the minimum and maximum values of the coordinate x for points of this segment.

The difference transport equation for the three-dimensional problem is constructed on the basis of the energy balance for an elementary rectangular parallelepiped, forming the coordinate planes: $(z_{i+1} + z_j)/2$, $(z_j + z_{j-1})/2$, $(y_{m+1,j} + y_{mj})/2$, $(y_{m,j} + y_{m-1,j})/2$, $(x_{i+1,mj} + x_{imj})/2$, $(x_{imj} + x_{i-1,mj})/2$.

The specific fluxes q_x'' and q_y'' through the surfaces $(x_{i+1,mj} + x_{imj})/2$ and $(y_{m+1,j} + y_{mj})/2$ are found from expressions similar to (4) and (9):

$$q_x'' = -\frac{\lambda_{i+1,mj} + \lambda_{imj}}{2}\,\frac{t^n_{i+1,mj} - t^n_{imj}}{\Delta x_{mj}},$$

$$q_y'' = -\frac{\lambda_{im+1,j} + \lambda_{imj}}{2}\,\frac{t^n_{im+1j} - t^n_{imj}}{\Delta y_m} - \bar{q}''_{xy}\,\frac{x_{im+1,j} - x_{imj}}{\Delta y_m},$$

$$\bar{q}''_{xy} = (1-\theta_q)\,\frac{q'_{ximj} + q''_{ximj}}{2} + \theta_q\,\frac{q'_{xim+1,j} + q''_{xim+1,j}}{2}.$$

The thermal flux q_z'' through the boundary $(z_{j+1} + z_j)/2$ is found in two steps. At the first step one finds the projection q_{zy}'' of the maximum thermal flux in the plane passing through the line (z_j, y_{mj}) and the point $(x_{i,m+1,j}, y_{m+1,j}, z_j)$ on to z0y-plane:

$$q''_{zy} = -\frac{\lambda_{imj+1} + \lambda_{imj}}{2}\,\frac{t^n_{imj+1} - t^n_{imj}}{\sqrt{\Delta z^2 + (y_{i,m+1} - y_{im})^2}} - \bar{q}_{xz}\,\frac{x_{imj+1} - x_{imj}}{\sqrt{\Delta z^2 + (y_{im+1} - y_{im})^2}},$$

$$\bar{q}_{xz} = (1-\Theta_q)\,\frac{q'_{ximj} + q''_{ximj}}{2} + \theta_q\,\frac{q'_{ximj+1} + q''_{ximj+1}}{2}.$$

At the second step one finds the projection of the maximum flux in the z0y-plane on to the z-axis:

$$q''_z = q''_{zy}\,\frac{\sqrt{\Delta z^2 + (y_{im+1} - y_{im})^2}}{\Delta z} - \bar{q}_{yz}\,\frac{y_{m+1,j} - y_{mj}}{\Delta z} =$$

$$= -\frac{\lambda_{imj+1} + \lambda_{imj}}{2}\,\frac{t_{imj+i} - t_{imj}}{\Delta z} - \bar{q}_{xz}\,\frac{x_{imj+1} - x_{imj}}{\Delta z} - \bar{q}_{yz}\,\frac{y_{m+1,j} - y_{mj}}{\Delta z}.$$

The specific flux q_z' through the boundary $(z_j + z_{j-1})/2$ is found similarly:

$$q'_z = -\frac{\lambda_{imj} + \lambda_{imj-1}}{2}\,\frac{t_{imj} - t_{imj-1}}{\Delta z} - \bar{q}_{xz}\,\frac{x_{imj} - x_{imj-1}}{\Delta z} - \bar{q}_{yz}\,\frac{y_{mj} - y_{m-1,j}}{\Delta z}.$$

The energy balance equation for the elementary volume considered is

$$c\rho\left[(1+\theta)\,\frac{t^{n+1}_{imj} - t^n_{imj}}{\Delta\tau^{n+1}} - \theta\,\frac{t^n_{imj} - t^{n-1}_{imj}}{\Delta\tau^n}\right] = \frac{q'_x - q''_x}{\Delta x_{mj}} + \frac{q'_y - q''_y}{\Delta y_m} + \frac{q'_z - q''_z}{\Delta z}.$$

By replacing the left hand sides of Eqs. (11) and (13) by the expression $c\rho(t_{imj}^n - t^{n-1}_{imj})/\Delta\tau$ we reach implicit heat transfer equations for canonical shape elements on a non-uniform grid. These equations are absolutely stable, but the solution algorithm is substantially more complicated for this replacement.

The solution method described was tested numerically by solving a number of heat conduction problems for bodies with curvilinear boundaries, in particular for an infinite cylinder of radius R with heat exchange boundary conditions of the first and third kind in a Cartesian coordiante system. The solution is taken on 1/4 of a circle on the difference grid: $y_m = m\Delta y$, $m = 0, 1, \ldots, M$, $\Delta y = R/M$; $x_{mi} = i\Delta x_m$, $i = 0, 1, \ldots, I$, $\Delta x_m = x''/I$, $x_M'' = R\sqrt{1 - (y_m/R)^2}$; $\tau_n = n\Delta\tau$, $n = 0, 1, \ldots$, $\Delta\tau = \text{const}$. The extent of nonuniformity of the grid selected is quite substantial, since the step Δx_m varies from 0 at y = R to R/I at y = 0. The initial temperature of the cylinder is $t_0 = \text{const}$. Table 1 provides results of comparing the relative temperature $\vartheta(x, y) = \dfrac{t - t_b}{t_0 - t_b}$, determined on the basis of numerical

TABLE 1. Comparison of Numerical [$\vartheta(x, y)$] and Analytic [$\vartheta_a(x, y)$] Solutions of Heat Conduction Problem in an Infinite Cylinder with First Kind Boundary Conditions

Fo	$\vartheta(0, 0)$	$\vartheta_a(0, 0)$	$\vartheta(0, R/2)$	$\vartheta(R/2, 0)$	$\vartheta_a(0, R/2)$
0,01	1,0	1,0	0,998	0,998	0,999
0,02	1,0	1,0	0,982	0,980	0,980
0,03	0,994	0,9995	0,948	0,945	0,940
0,04	0,9964	0,9963	0,903	0,900	0,884
0,05	0,9887	0,9871	0,857	0,852	0,837
0,06	0,9750	0,9705	0,804	0,808	0,784
0,07	0,9550	0,9470	0,755	0,763	0,730
0,08	0,9293	0,9177	0,709	0,720	0,683
0,09	0,8991	0,8844	0,666	0,679	0,643
0,1	0,8652	0,8484	0,626	0,641	0,618
0,2	0,5182	0,5015	0,342	0,356	0,339
0,3	0,2913	0,2825	0,190	0,194	0,188
0,4	0,1621	0,1585	0,105	0,108	0,103
0,5	0,0904	0,0887	0,058	0,061	0,066
0,6	0,0502	0,0499	0,033	0,034	0,032
0,7	0,0280	0,0280	0,018	0,019	0,018
0,8	0,0156	0,0157	0,010	0,011	0,0010
0,9	0,0087	0,0088	0,0057	0,0059	0,0055
1,0	0,0048	0,0049	0,0032	0,0033	0,0031

TABLE 2. Comparison of Numerical [$\vartheta(x, y)$] and Analytic [$\vartheta_a(x, y)$] Solutions of Heat Conduction Problem in an Infinite Cylinder with Third Kind Boundary Conditions for Bi = 0.1

Fo	$\vartheta(0, 0)$	$\vartheta_a(0, 0)$	$\vartheta(0, R)$	$\vartheta(R, 0)$	$\vartheta_a(0, R)$
0,05	0,999	1,00	0,968	0,9696	0,975
0,1	0,997	0,998	0,956	0,9584	0,960
0,5	0,928	0,929	0,879	0,884	0,881
1,0	0,839	0,841	0,795	0,800	0,804
2,0	0,692	0,694	0,649	0,654	0,660
3,0	0,561	0,565	0,530	0,534	0,539
4,0	0,4579	0,463	0,434	0,437	0,441
5,0	0,3739	0,380	0,355	0,357	0,3647
6,0	0,306	0,317	0,292	0,293	0,300
7,0	0,251	0,259	0,237	0,239	0,249
8,0	0,205	0,208	0,195	0,196	0,207
9,0	0,172	0,178	0,159	0,160	0,164
10	0,137	0,141	0,132	0,133	0,136
15	0,051	0,053	0,049	0,050	0,0523
20	0,019	0,020	0,017	0,017	0,018

and analytic [6] solutions, at the cylinder axis (x = 0, y = 0) and at the points (x = 0, y = R/2) and (x = R/2, y = 0) for various values of the Fourier number $Fo = \lambda\tau/c\rho R^2$ for heat exchange conditions of the first kind and constant temperature t_b at the external boundary.

Table 2 provides results of numerical and analytic [6] determinations of the relative temperature $\vartheta(x, y) = \frac{t - t_0}{t_s - t_0}$ on the cylinder axis and at the points (x = 0, y = R) and (x = R, y = 0) for various values of the Fourier number for heat exchange conditions of the third kind with Biot number $Bi = \alpha R/\lambda = 0.1$ and constant temperature of the surrounding medium t_s. As seen from the table, the data of the numerical solution on a nonuniform grid with I = M = 10 agree quite well with the analytic solutions.

The results of the numerical experiments indicate the effectiveness of the canonical element method. Transition from one body geometry to another requires changes of a few commands only in the corresponding computer program, related to the functional description of the boundary surface. This fact is favorable for creating, on the basis of the method described, universal program packages for simulating heat-transfer processes in complicated elements of contemporary technology.

LITERATURE CITED

1. N. I. Nikitenko, Theory of Heat and Mass Transfer [in Russian], Kiev (1983).
2. A. A. Samarskii, Theory of Difference Algorithms [in Russian], Moscow (1977).
3. L. J. Segerlind, Applied Finite Element Analysis, 2nd ed., Wiley, New York (1984).
4. N. I. Nikitenko and Yu. N. Nikitenko, Promyshl. Teplotekh., 3, No. 6, 7-12 (1983).
5. N. I. Nikitenko, Associated and Inverse Problems in Heat and Mass Transfer [in Russian], Kiev (1988).
6. A. V. Lykov, Analytical Heat Diffusion Theory, Academic Press, New York (1968).

SOLUTION OF THE EXTERIOR AND INTERIOR DIRICHLET PROBLEM OF POTENTIAL THEORY IN A MULTIPLY CONNECTED DOMAIN

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Through use of a complement to the solution of a heat conduction boundary value problem of Dirichlet type (presented classically in the form of a double layer potential) we obtain by means of simple sources singular integral equations (SIE) for exterior and interior multiply connected domains. Algorithms and a computer program were developed to obtain a numerical solution of the SIE.

In considering thermal problems of Dirichlet type by method of the potential (temperature T is a harmonic function and is subject to the equation of Laplace) two traditional methods are employed: classical (nondirect) and nonclassical (direct).

The classical method consists in seeking a solution in the form of a double layer potential:

$$T = \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y. \tag{1}$$

Its limiting value at points of boundary S of domain V is equated to the given function and we obtain the following integral equation:

$$T(x_S) = 2\pi\varkappa(x_S)\eta + \text{v. p.} \oint_S \varkappa(y) \frac{\cos\varphi}{r^2} dS_y. \tag{2}$$

Here $T(x_S)$ is a given value of the function on boundary S of domain V; κ is the density of the double layer potential; φ is the angle between vector $r = |y - x|$ and the exterior normal n_y to S at the integration point y; $\eta = 1$ for the inner limit; $\eta = 0$ for the direct value; and $\eta = -1$ for the outer limit; v. p. indicates principal value of the Cauchy-type integral.

This method is used, however, only in the case of an interior simply connected domain [1]. For an exterior domain (even a simply connected one) it is not a suitable method. Actually the double layer potential can only represent the temperature of the exterior domain partially. If the temperature is split up into two components, a constant component and a variable component, $T = T^{(m)} + T^{(v)}$, where $T^{(m)}$ is the mean value, the influence of the mean temperature $T^{(m)}$ is then not taken into account by the double layer potential. In addition, for a simply connected exterior domain even a variable temperature field cannot be represented by a double layer potential if the sources are distributed uniformly over the boundary surface ($\kappa(y)$ = const):

$$T = \varkappa \oint_S d\omega_S = \varkappa \oint_S \frac{\cos\varphi}{r^2} dS = \varkappa\omega_S \equiv 0 \tag{3}$$

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